

# An additive variant of Tsallis generalized entropy

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This is an analysis of the additivity of the entropy of thermodynamical systems with finite heat baths. It is presented an expression for the physical entropy of weakly interacting ergodic systems, and it is shown that it is valid for both the microcanonical (constant energy), the Boltzmann-Gibbs canonical (infinite heat bath) and the Tsallis (finite heat bath) ensembles. This physical entropy may be written as a variant of Tsallis entropy. It becomes an additive function after a suitable choice of additive constants, in a procedure reminiscent to the solution presented by Gibbs to the paradox bearing his name.

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## I. INTRODUCTION

The non-extensivity is one of the major features of Tsallis entropy and perhaps the reason for most of the criticism it receives. This property was recognized in the very first paper [1] where the generalized entropy has been proposed in its discrete form

$$S_q(X) = \frac{1 - \sum_{i=1}^N [p(X = x_i)]^q}{q - 1}, \quad (1)$$

where  $q$  is a real constant and  $p(X = x_i)$  is the probability distribution of a discrete random variable  $X$  that takes value on  $\{x_1, \dots, x_N\}$ . Tsallis [1] obtained the *non-extensive* (non-additive) relation

$$S_q(X, Y) = S_q(X) + S_q(Y) - (q - 1)S_q(X)S_q(Y) \quad (2)$$

computing the entropy of the joint distribution of two independently distributed random variables  $X$  and  $Y$ . Since then, the words “extensive” and “additive” have been loosely used as synonymous in the literature about Tsallis statistics [2]. However, following Touchette’s [2] guidelines for the usage of these terms, the property expressed by Eq. (2) is called here “non-additivity”.

In a recent paper, Abe [3] has shown that, for composite systems resulting from the joint observation of two independently distributed sub-systems, the “physical entropy” is additive at the macroscopic level, even in the context of Tsallis statistics, where by “physical entropy” he means the function (Rényi entropy)

$$S^* = \frac{\ln(1 + (1 - q)S_q)}{1 - q} = \frac{\ln(\sum_{i=1}^N [p(X = x_i)]^q)}{1 - q}, \quad (3)$$

where  $S_q$  is the Tsallis entropy.

Power-law (Tsallis) canonical distributions have been connected to the finitude of the dimension of the phase space of the heat bath in [4], [5] and [9].

The issue of additivity within Tsallis statistics has been addressed in the scientific literature up to now assuming *the joint observation of independent random variables* as done in the derivation of Eq. (2). Regardless of the validity of this rule for non-interacting systems and for “weakly interacting systems” in the thermodynamic limit, the same can not be generally applied for finite thermodynamic systems. We say that two systems are in weak interaction when the Hamiltonian of the composite system is just the sum of the Hamiltonians of its sub-components,  $H_{1+2} = H_1 + H_2$ . As a consequence, the internal energy ( $U = \int H \rho d\mathbf{q} d\mathbf{p}$ ) of such a system in an equilibrium state  $\rho$  is an additive variable.

Consider, for instance, the observation of the energy. When one joins (through weak interaction) two independent sub-systems,  $A_1$  and  $A_2$ , the distribution of the energy of the composite system  $A$ , ( $E_A = E_{A_1} + E_{A_2}$ ), is given by the convolution of the sub-component’s distributions,  $\rho(E_A) = \rho(E_{A_1}) \otimes \rho(E_{A_2})$ , which is generally different

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from the distribution of the joint observation of  $E_{A_1}$  and  $E_{A_2}$ ,  $\rho(E_{A_1}, E_{A_2}) = \rho(E_{A_1})\rho(E_{A_2})$ . In other words, when observing  $E_A$ , it is not necessary to know how much of it comes from system  $A_1$  or  $A_2$ , but only that the sum of these contributions adds up to the observed quantity. In addition to this, even the range of the energy value of each sub-system changes upon the composition: If each separate sub-system initially has internal energy equal to  $E_0$ , after their merging the composite system will have energy equal to  $2E_0$  and the energy of each sub-system will now fluctuate in the interval  $[0, 2E_0]$ . Therefore, the joint distribution of the simultaneous observation of two independent systems  $A_1$  and  $A_2$  does not represent the composite thermodynamic system  $A_1 + A_2$ . Since the distribution of a sum of independent random variables is a “convolution” of the distributions of the component variables, the additivity of the energy implies that the correct distribution for the composite system shall be obtained through a convolution operation. Only in the thermodynamic limit, when the sub-systems become infinite, the resulting distribution will factor into the product of the component distributions. Moreover, it can be rigorously proof that for systems in weak interaction the Boltzmann-Gibbs canonical distribution (exponential) is the only one for which the convolution of the components equals their product.

We will derive a *physical entropy function*  $S$  that is additive at the equilibrium distribution even for finite systems. For a given system  $A$  resulting from a *weak combination* of two sub-systems  $A_1$  and  $A_2$ , (all of them in equilibrium), the entropy function  $S$  satisfies the *additive* rule  $S(A) = S(A_1) + S(A_2)$ . We will explicitly evaluate  $S$  for canonical ergodic systems. This analysis will be based on the canonical ensemble theory of ergodic systems as presented by Kinchin [6], whose basic elements we recollect in Section II. The present work is a follow up of the study started in [5] where it was presented a one-to-one connection between the canonical distribution and the form of the entropy, and the Tsallis entropy was derived from first principles. This alternative approach for Tsallis statistics has been already used by Andrade et al. [7] to formulate a molecular dynamics scheme for Hamiltonian systems exhibiting power-law distributions, generalizing the extended Hamiltonian approach presented by Nosé [8]. This setup was also used by Adib et al. [9] to analyse the statistical behavior of systems with homogenous Hamiltonians.

## II. ERGODIC SYSTEMS

For a generic system with Hamiltonian  $H = H(\mathbf{q}, \mathbf{p})$  and phase space  $\Xi$ , let  $V(E)$  denote the volume of the region of  $\Xi$  where  $H(\mathbf{q}, \mathbf{p}) \leq E$ ,  $\Omega(E) = \partial V(E)/\partial E$  be the *structure function* [6], and  $\beta(E)$  be the function

$$\beta(E) = \frac{d}{dE} \ln \Omega(E). \quad (4)$$

The distribution on the surface of constant energy ( $\Sigma(E)$ ) that is invariant under the motion of the system is called *microcanonical* and has density given by (see Chapter II of [6])

$$\rho^{(\text{mc})}(\mathbf{q}, \mathbf{p}) = \frac{1}{\Omega(E) \|\nabla H(\mathbf{q}, \mathbf{p})\|}, \quad (5)$$

where  $\|\nabla H(\mathbf{q}, \mathbf{p})\|$  is the norm of the gradient of the Hamiltonian  $H(\mathbf{q}, \mathbf{p})$ .

For a system  $A$  which is the *weak combination* of two sub-systems  $A_1$  and  $A_2$  (denoted by  $A = A_1 \uplus A_2$ ), we have that  $H(\mathbf{q}, \mathbf{p}) = H_1(\mathbf{q}_1, \mathbf{p}_1) + H_2(\mathbf{q}_2, \mathbf{p}_2)$  with no interaction term between  $A_1$  and  $A_2$  in the Hamiltonian, i.e., the energies of the sub-systems are additive. In that case the canonical distribution in  $\Xi_1$ , when the total energy of the composite system  $A$  is equal to  $E$ , is given by the density function [6]

$$\rho_1(\mathbf{q}, \mathbf{p}) = \frac{\Omega_2(E - H_1(\mathbf{q}, \mathbf{p}))}{\Omega(E)}. \quad (6)$$

The structure function of  $A$  satisfies the convolution relation

$$\Omega(E) = \int_0^E \Omega_1(\xi) \Omega_2(E - \xi) d\xi. \quad (7)$$

Observe that the microcanonical distribution on  $\Sigma_1(E_1)$  is the conditional distribution of  $\rho_1$  obtained by fixing the value of  $H_1(\mathbf{q}, \mathbf{p}) = E_1$ , which results in

$$\rho_1^{(\text{mc})}(\mathbf{q}, \mathbf{p}) = \frac{1}{\Omega_1(E_1) \|\nabla H_1(\mathbf{q}, \mathbf{p})\|}. \quad (8)$$

It has been shown in [5] that the form of the canonical  $\rho_1$  depends on the value of the parameter  $q_2$ , which is related to  $\beta_2(E)$  through the expression

$$\frac{d}{dE} \left( \frac{1}{\beta_2(E)} \right) = q_2 - 1. \quad (9)$$

When  $q_2 = 1$ ,  $\rho_1$  is an exponential (Boltzmann-Gibbs) and, when  $q_2 \neq 1$ ,  $\rho_1$  is a power-law (Tsallis) distribution. Identifying system  $A_2$  with the heat bath  $\mathcal{B}$  and integrating Eq. (9) with the initial condition  $\Omega_2(0) = 0$  we get that

$$\Omega_2(E) = C_0 E^{1/(q_2-1)}, \quad (10)$$

where  $C_0$  is a constant, and consequently

$$\beta_2(E) = \frac{1}{(q_2 - 1)E}. \quad (11)$$

We are going to consider that the elementary systems appearing from now on have structure functions in the form  $\Omega(E) = CE^{1/(q-1)}$ , with specific constants  $C$  and  $q$  for each one. The parameter  $q$ , which is a signature of the system's phase space geometry, may vary from system to system. Indeed, Almeida et al. [10] have showed that the weak composition of two systems  $\mathcal{O}$  and  $\mathcal{B}$ , with parameters  $q_{\mathcal{O}}$  and  $q_{\mathcal{B}}$ , respectively, results in a system  $A$  with parameter  $q_A$  given by

$$\frac{q_A}{q_A - 1} = \frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} + \frac{q_{\mathcal{B}}}{q_{\mathcal{B}} - 1}, \quad (12)$$

and that the temperature definition obeying the theorem of equipartition of energy leads to the expressions

$$kT = \left( \frac{q_A - 1}{q_A} \right) E_A \quad (13)$$

$$= \left( \frac{q_{\mathcal{O}} - 1}{q_{\mathcal{O}}} \right) \langle H_{\mathcal{O}} \rangle \quad (14)$$

$$= \left( \frac{q_{\mathcal{B}} - 1}{q_{\mathcal{B}}} \right) \langle H_{\mathcal{B}} \rangle. \quad (15)$$

Let's consider a weak combination of independent canonical sub-systems  $A_i$ , each one having an observable sub-system  $\mathcal{O}_i = \mathcal{O}(A_i)$  and a heat bath  $\mathcal{B}_i = \mathcal{B}(A_i)$ . The observable and the heat bath of the composite system  $A$  are, respectively, the weak combinations of the observables and of the heat baths of the sub-systems, i.e.,  $A = \uplus_i A_i$ ,  $\mathcal{O}(A) = \uplus_i \mathcal{O}(A_i)$ , and  $\mathcal{B}(A) = \uplus_i \mathcal{B}(A_i)$ . Identifying the observable part  $\mathcal{O}(= \mathcal{O}(A))$  and the heat bath  $\mathcal{B}(= \mathcal{B}(A))$ , respectively, with the systems  $A_1$  and  $A_2$  in Eq. (6), we write the canonical distribution of  $A$  as

$$\rho_{\mathcal{O}}(H_{\mathcal{O}}) = \frac{\Omega_{\mathcal{B}}(E_A - H_{\mathcal{O}})}{\Omega_A(E_A)}, \quad (16)$$

where  $E_A$  is the total energy of  $A$ ,  $H_{\mathcal{O}}$  is the energy of its observable sub-system, and  $\Omega_A$  and  $\Omega_{\mathcal{B}}$  are the structure functions of the total system  $A$  and of its heat bath  $\mathcal{B}$ , respectively. Writing  $\rho_{\mathcal{O}}$  in the usual Tsallis density form

$$\rho(H_{\mathcal{O}}) = \rho_{\mathcal{O}}(H_{\mathcal{O}}) = \rho_0 [1 - (q_{\mathcal{B}} - 1)\beta H_{\mathcal{O}}]^{1/(q_{\mathcal{B}}-1)}, \quad (17)$$

we have that

$$\rho_0 = \frac{\Omega_{\mathcal{B}}(E_A)}{\Omega_A(E_A)} \quad (18)$$

and

$$\beta = \beta_{\mathcal{B}}(E_A) = \frac{1}{(q_{\mathcal{B}} - 1)E_A}. \quad (19)$$

Moreover, the equivalent to Eq. (12) reads as

$$\frac{q_A}{q_A - 1} = \frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} + \frac{q_{\mathcal{B}}}{q_{\mathcal{B}} - 1}, \quad (20)$$

where  $q_A$ ,  $q_{\mathcal{O}}$  and  $q_{\mathcal{B}}$  are the parameters  $q$  of  $A$ ,  $\mathcal{O}$  and  $\mathcal{B}$ , respectively.

Applying Eq. (12) recursively to  $A$ ,  $\mathcal{O}$  and  $\mathcal{B}$  we get the relations

$$\frac{q_A}{q_A - 1} = \sum_i \frac{q_{A_i}}{q_{A_i} - 1} \quad (21)$$

$$\frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} = \sum_i \frac{q_{\mathcal{O}_i}}{q_{\mathcal{O}_i} - 1} \quad (22)$$

and

$$\frac{q_{\mathcal{B}}}{q_{\mathcal{B}} - 1} = \sum_i \frac{q_{\mathcal{B}_i}}{q_{\mathcal{B}_i} - 1}. \quad (23)$$

### III. PHYSICAL ENTROPY

For a system with Hamiltonian  $H$  and probability density function  $\rho$  let  $U$  be the internal energy,  $W$  be the work of the external forces and  $Q$  be the heat supplied to the system. In terms of the microscopic functions we have that

$$U = \int_{\Xi} H \rho d\mathbf{q} d\mathbf{p}. \quad (24)$$

The first law of thermodynamics states that

$$\delta U = \delta W + \delta Q. \quad (25)$$

Computing the variation  $\delta U$  from its microscopic definition, Eq. (24), we get that

$$\delta U = \int_{\Xi} (\delta H) \rho d\mathbf{q} d\mathbf{p} + \int_{\Xi} H (\delta \rho) d\mathbf{q} d\mathbf{p}. \quad (26)$$

The first term on the right hand side of this equation is the variation of the work  $\delta W$  and so we get that

$$\delta Q = \int_{\Xi} H (\delta \rho) d\mathbf{q} d\mathbf{p}. \quad (27)$$

The second law of thermodynamics states that there are an exact differential functions  $S$ , called entropies, such that

$$\delta S = \frac{1}{T} \delta Q, \quad (28)$$

where  $T$ , the integrating factor, is a temperature scale.

As pointed out in a previous work [5], the functional form of the physical entropy is determined by the canonical distribution: Given a canonical density function in  $\Xi$ ,  $\rho = f(H)$ , with a non-negative monotonic function  $f(x)$  defined for  $x \geq 0$ , the functions (entropies) that satisfy the second law of thermodynamics are given by

$$S(\rho) = \int_{\Xi} F(\rho) d\mathbf{q} d\mathbf{p} \quad (29)$$

with

$$F(\rho) = \frac{1}{T} \int_0^\rho f^{-1}(\xi) d\xi + \psi, \quad (30)$$

where  $\psi$  is an arbitrary constant. As will be seen later, it is convenient to choose  $\psi = -k$ , where  $k$  is the Boltzmann constant.

Applying the above procedure to the Tsallis density function

$$\rho(H) = \rho_0 [1 - (q_{\mathcal{B}} - 1)\beta H]^{1/(q_{\mathcal{B}} - 1)}, \quad (31)$$

where  $(q_B - 1)\beta = 1/E$  is the inverse of the total energy of the system, considering that

$$\rho_0 = \rho(0) = \left\{ \int_{\Xi} [1 - (q_B - 1)\beta H]^{1/(q_B - 1)} d\mathbf{q}d\mathbf{p} \right\}^{-1}, \quad (32)$$

is constant, we get the function

$$\mathcal{E}(\rho) = \frac{1}{T (q_B - 1)\beta} \left[ 1 - \frac{\rho_0}{q_B} \int_{\Xi} \left( \frac{\rho}{\rho_0} \right)^{q_B} d\mathbf{q}d\mathbf{p} \right] - k, \quad (33)$$

which is not the entropy yet, because of the imposed condition that  $\rho_0$  be constant. Collecting and rearranging terms we get the alternative expression

$$\begin{aligned} \mathcal{E}(\rho) = \frac{1}{q_B T \beta} \left\{ \frac{\rho_0}{q_B - 1} \int_{\Xi} \left[ \left( \frac{\rho}{\rho_0} \right) - \left( \frac{\rho}{\rho_0} \right)^{q_B} \right] d\mathbf{q}d\mathbf{p} \right\} \\ + \left( \frac{1}{q_B T \beta} - k \right). \end{aligned} \quad (34)$$

This expression may be written also as

$$\mathcal{E}(\rho) = \left( \frac{1}{q_B T \beta} \right) \rho_0 S_{\text{TS}}(\rho/\rho_0) + \left( \frac{1}{q_B T \beta} - k \right), \quad (35)$$

where

$$S_{\text{TS}}(\rho/\rho_0) = \frac{1}{q_B - 1} \int_{\Xi} \left[ \left( \frac{\rho}{\rho_0} \right) - \left( \frac{\rho}{\rho_0} \right)^{q_B} \right] d\mathbf{q}d\mathbf{p} \quad (36)$$

is the Tsallis entropy evaluated at the ratio  $\rho/\rho_0$ . Observe in Eq. (35) the splitting of the roles played by the density function form,  $\rho/\rho_0$ , and by the normalization value  $\rho_0$ .

Now, considering that both  $\rho$  and  $\rho_0$  vary along the process, the evaluation of  $\delta\mathcal{E}$  will produce the sought expression for the entropy. From Eq. (33) we have that

$$\begin{aligned} \delta\mathcal{E} = \frac{1}{T (q_B - 1)\beta} \left[ - \int_{\Xi} \left( \frac{\rho}{\rho_0} \right)^{q_B - 1} (\delta\rho) d\mathbf{q}d\mathbf{p} + \right. \\ \left. (\delta\rho_0) \frac{q_B - 1}{q_B} \int_{\Xi} \left( \frac{\rho}{\rho_0} \right)^{q_B} d\mathbf{q}d\mathbf{p} \right]. \end{aligned} \quad (37)$$

Substituting the expressions for  $\rho$  and  $\rho_0$ , and considering the temperature expressions Eqs. (13)-(15) we get that

$$\delta\mathcal{E} = \frac{1}{T} \int_{\Xi} H(\delta\rho) d\mathbf{q}d\mathbf{p} + k\delta(\ln \rho_0) = \frac{1}{T} \delta Q + k\delta(\ln \rho_0), \quad (38)$$

and from it we conclude that

$$\delta S = \frac{1}{T} \delta Q = \delta\mathcal{E} - k\delta(\ln \rho_0), \quad (39)$$

and so the generic entropies are

$$S(\rho) = \mathcal{E}(\rho) - k \ln \rho_0 + C, \quad (40)$$

where  $C$  is an arbitrary constant.

For a fixed value of  $D = q_B T \beta$ , the maximization of  $S(\rho)$  under the conditions

$$\int \rho d\mathbf{q}d\mathbf{p} = 1, \quad (41)$$

$$\int H \rho d\mathbf{q}d\mathbf{p} = U, \quad (42)$$

and

$$\rho_0 = \rho(0), \quad (43)$$

results in the Tsallis density function  $\rho(H)$  given by Eqs. (31) and (32).

### A. Additivity of $\mathcal{E}$

Let's compute the explicit value of  $\mathcal{E}_A(\rho)$  for a system  $A(= \mathcal{O} \uplus \mathcal{B})$  whose canonical distribution on  $\Xi_{\mathcal{O}}$ ,  $\rho(H_{\mathcal{O}})$ , is given in Eq. (17). From a direct integration we get

$$\rho_0 \int_{\Xi_{\mathcal{O}}} \left( \frac{\rho}{\rho_0} \right)^{q_{\mathcal{B}}} d\mathbf{q} d\mathbf{p} = \left[ 1 - \frac{\langle H_{\mathcal{O}} \rangle}{E_A} \right], \quad (44)$$

where we adopt the notation  $\langle f \rangle = \int_{\Xi_{\mathcal{O}}} f \rho d\mathbf{q} d\mathbf{p}$  for the average of a generic function  $f$ .

From Eqs. (19) and (13) we get that

$$\frac{1}{T (q_{\mathcal{B}} - 1)\beta} = k \left( \frac{q_A}{q_A - 1} \right). \quad (45)$$

Substituting Eqs. (44) and (45) in Eq. (33) and using Eq. (20) one gets that

$$\mathcal{E}_A(\rho) = k \left( \frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} \right). \quad (46)$$

Using the combination rule for the  $q_{\mathcal{O}}$ 's, Eq. (22), we observe that under *weak combination* of sub-systems the function  $\mathcal{E}_A$  is additive. In general, if  $A$  is a (weak) combination of  $N$  sub-systems  $A_i$ ,  $i = 1, \dots, N$ , ( $A = A_1 \uplus A_2 \uplus \dots \uplus A_N$ ), all at the same temperature  $T$ , then

$$\mathcal{E}_A(\rho) = \sum_{i=1}^N \mathcal{E}_{A_i}(\rho_i). \quad (47)$$

Notice that  $\mathcal{E}_A(\rho)$  does not depend on the parameter  $q_{\mathcal{B}}$ , hence Eq. (46) is valid in the limit case  $q_{\mathcal{B}} \rightarrow 1$  (Boltzmann-Gibbs), which can be also verified through a direct evaluation of Eq. (56) with  $\rho = \rho_0 \exp(-\beta H)$ . Observe that Eq. (14) together with Eq. (46) leads to the relation between  $\mathcal{E}_A$ , the internal energy  $U = \langle H_{\mathcal{O}} \rangle$  and the temperature  $T$ ,

$$\mathcal{E}_A(\rho) = \frac{U}{T}. \quad (48)$$

Substituting the alternative forms of  $\mathcal{E}(\rho)$  into Eq. (40) we obtain

$$S(\rho) = k \left( \frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} \right) - k \ln \rho_0 + C \quad (49)$$

and

$$S(\rho) = \frac{U}{T} - k \ln \rho_0 + C. \quad (50)$$

This last expression was also obtained through a different route by Potiguar and Costa [11].

Notice that the constants  $q_{\mathcal{B}}$  and  $\beta$  appearing in Eq. (34) are associated with the heat bath and that

$$\frac{1}{q_{\mathcal{B}} T \beta} = k \left( \frac{q_A}{q_A - 1} \right) \left( \frac{q_{\mathcal{B}} - 1}{q_{\mathcal{B}}} \right) \quad (51)$$

$$= k \left( \frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} \right) \left( \frac{q_{\mathcal{B}} - 1}{q_{\mathcal{B}}} \right) + k \quad (52)$$

is determined by the constants  $q_{\mathcal{O}}$  and  $q_{\mathcal{B}}$ . From this last equation, we see that the commonly used relation  $kT\beta = 1$  is valid only in the limit  $q_{\mathcal{B}} \rightarrow 1$ , i.e., in the Boltzmann-Gibbs statistics. This fact has been overseen by some authors who inadvertently adopted  $kT = 1/\beta$  within Tsallis statistics, which might have led to misleading results.

### B. Microcanonical ensemble

The microcanonical ensemble is represented by a constant energy  $H_{\mathcal{O}}$  and its density function in this surface  $\Sigma_{\mathcal{O}}$  is

$$\rho^{(\text{mc})}(H_{\mathcal{O}}) = \frac{1}{\Omega_{\mathcal{O}}(H_{\mathcal{O}}) \|\nabla H_{\mathcal{O}}\|}. \quad (53)$$

Hence, we may consider that in Eq. (53)  $\rho_0 = (\Omega_{\mathcal{O}})^{-1}$  and obtain from Eq. (49) that

$$S(\rho^{(\text{mc})}) = k \ln [\Omega_{\mathcal{O}}(H_{\mathcal{O}})] + \left[ k \left( \frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} \right) + C \right], \quad (54)$$

which is the Boltzmann entropy for the microcanonical ensemble plus constants.

### C. Boltzmann-Gibbs canonical ensemble

The Boltzmann-Gibbs canonical ensemble considers an infinite heat bath, which is represented by the limit condition  $q_{\mathcal{B}} \rightarrow 1$  keeping  $\beta$  fixed. Taking this limit in Eq. (17) and in either of the Eqs. (33) or (34) we arrive at

$$\rho(H_{\mathcal{O}}) = \rho_0 \exp(-\beta H_{\mathcal{O}}) \quad (55)$$

and

$$\mathcal{E}(\rho) = \frac{1}{T\beta} \left\{ 1 - \int_{\Xi} \rho \ln(\rho) d\mathbf{q} d\mathbf{p} + \ln(\rho_0) \right\} - k, \quad (56)$$

from which, considering also that in the limit  $q_{\mathcal{B}} \rightarrow 1$  we have  $1/(T\beta) = k$  (see Eq. (52) below), we get that

$$S = \mathcal{E}(\rho) - k \ln(\rho_0) + C = -k \int_{\Xi} \rho \ln(\rho) d\mathbf{q} d\mathbf{p} + C \quad (57)$$

is the Boltzmann-Gibbs-Shannon entropy plus an arbitrary constant  $C$ .

### D. Additivity of $k \ln \rho_0$

Let's consider that the system  $A$  is the weak combination of  $n$  identical sub-systems  $A_i$ ,  $i = 1, \dots, n$ , which by their turn are the weak combination of an observable  $\mathcal{O}_i$  and a heat bath  $\mathcal{B}_i$ . Let  $q$ ,  $q_1$  and  $q_2$  be the  $q$ -parameters of the  $A_i$ ,  $\mathcal{O}_i$  and  $\mathcal{B}_i$ , respectively. Therefore, we have that the parameters for the system  $A$ , its observable part  $\mathcal{O}$ , and its heat bath  $\mathcal{B}$  are respectively  $q_A$ ,  $q_{\mathcal{O}}$  and  $q_{\mathcal{B}}$ , and they satisfy the relations of Eqs. (21)-(23), i.e.,

$$\frac{q_A}{q_A - 1} = \sum_i \frac{q_{A_i}}{q_{A_i} - 1} = n \frac{q}{q - 1} \quad (58)$$

$$\frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1} = \sum_i \frac{q_{\mathcal{O}_i}}{q_{\mathcal{O}_i} - 1} = n \frac{q_1}{q_1 - 1} \quad (59)$$

and

$$\frac{q_{\mathcal{B}}}{q_{\mathcal{B}} - 1} = \sum_i \frac{q_{\mathcal{B}_i}}{q_{\mathcal{B}_i} - 1} = n \frac{q_2}{q_2 - 1}. \quad (60)$$

Computing the structure function of the system  $A$  using Eq. (7) we obtain

$$\Omega_A(E_A) = \Omega_{\mathcal{O}}(E_A) \Omega_{\mathcal{B}}(E_A) E_A \frac{\Gamma\left(\frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1}\right) \Gamma\left(\frac{q_{\mathcal{B}}}{q_{\mathcal{B}} - 1}\right)}{\Gamma\left(\frac{q_A}{q_A - 1}\right)} \quad (61)$$

Therefore,

$$\rho_0 = \frac{\Omega_{\mathcal{B}}(E_A)}{\Omega_A(E_A)} = \frac{\Gamma\left(\frac{q_A}{q_A - 1}\right)}{\Omega_{\mathcal{O}}(E_A) E_A \Gamma\left(\frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1}\right) \Gamma\left(\frac{q_{\mathcal{B}}}{q_{\mathcal{B}} - 1}\right)} \quad (62)$$

Recursively applying Eq. (7) to compute  $\Omega_{\mathcal{O}}(E_A)$  we get that

$$\Omega_{\mathcal{O}}(E_A) E_A \Gamma\left(\frac{q_{\mathcal{O}}}{q_{\mathcal{O}} - 1}\right) = \left[ C_0 \Gamma\left(\frac{q_1}{q_1 - 1}\right) E_A^{\frac{q_1}{q_1 - 1}} \right]^n, \quad (63)$$

where  $C_0$  is the multiplicative constant appearing in the structure functions  $\Omega_{\mathcal{O}_i}$ . Plugging (63) into (62) we obtain

$$\rho_0 = \frac{\Gamma\left(\frac{q_A}{q_A-1}\right)}{\Gamma\left(\frac{q_B}{q_B-1}\right)} \left[ C_0 \Gamma\left(\frac{q_1}{q_1-1}\right) E_A^{\frac{q_1}{q_1-1}} \right]^{-n}. \quad (64)$$

>From Eq. (13) the total energy  $E_A$  in terms of the temperature and  $q$  is

$$E_A = kT \left( \frac{q_A}{q_A-1} \right) = nkT \left( \frac{q}{q-1} \right), \quad (65)$$

>From these two last equations we get that

$$-k \ln \rho_0 = k \ln \left[ \frac{\Gamma\left(\frac{nq_2}{q_2-1}\right)}{\Gamma\left(\frac{nq}{q-1}\right)} \right] \quad (66)$$

$$+ kn \left( \frac{q_1}{q_1-1} \right) \ln[n] \quad (67)$$

$$+ kn \left( \frac{q_1}{q_1-1} \right) \ln \left[ kT \left( \frac{q}{q-1} \right) \right] \quad (68)$$

$$+ kn \ln \left[ C_0 \Gamma\left(\frac{q_1}{q_1-1}\right) \right]. \quad (69)$$

The terms in Eqs. (68) and (69) are additive since they are linear in  $n$ , but the terms in (66) and (67) are non-additive. We can eliminate this non-additivity by taking the constant  $C$  appearing on Eq. (40) equal to the negative of the sum of all terms on the right hand side of Eqs. (66)- (69) that depend, besides of  $n$ , only on the parameters related to the smallest system sub-components, viz.,  $q$ ,  $q_1$ ,  $q_2$  and  $C_0$ . This adequate choice of the constant  $C$  is analogous to the solution presented by Gibbs to the paradox bearing his name. In this way, we are left solely with the only term depending on  $T$ ,

$$-k \ln \rho_0 + C = kn \left( \frac{q_1}{q_1-1} \right) \ln[T] = k \left( \frac{q_{\mathcal{O}}}{q_{\mathcal{O}}-1} \right) \ln[T]. \quad (70)$$

Combining Eqs. (40), (46) and (70) the resulting entropy function becomes

$$S = k \left( \frac{q_{\mathcal{O}}}{q_{\mathcal{O}}-1} \right) [1 + \ln[T]] = \frac{U}{T} [1 + \ln[T]]. \quad (71)$$

Since the temperature  $T$  is an intensive variable and the internal energy is additive, this entropy form is additive.

#### IV. CONCLUSIONS

We have seen that both the microcanonical and the classical canonical (Boltzmann-Gibbs) distributions can be obtained from the Tsallis distribution. We have obtained also an expression for the “physical entropy” of weak-interaction systems with finite heat baths ( $q_B \neq 1$ ) and showed that, for systems with constant energy, it yields the classical microcanonical Boltzmann entropy and that when the heat bath gets infinite ( $q_H \rightarrow 1$ ), it becomes the classical Boltzmann-Gibbs-Shannon entropy. Moreover, we saw that this “physical entropy” may be written as a variant of Tsallis entropy plus the product of the Boltzmann constant and the logarithm of the partition function. In addition to this, we saw that it may be turned into an additive variable by adding appropriate constants in a procedure reminiscent to what was done by Gibbs to overcome the Gibbs paradox of the microcanonical entropy for ideal gases.

Our results strength the physical meaning of the generalized entropy by showing that the generalized thermostatics fills the gap between the microcanonical and the Boltzmann-Gibbs canonical ensembles. These results are summarized in the following frame:

Ensemble	microcanonical	Tsallis	BG canonical
heat bath	—	finite	infinite
constraint	$H_{\mathcal{O}} = \text{constant}$	$q_B \neq 1$	$q_B \rightarrow 1$
density $\rho$	$\rho_0 [  \nabla H  ]^{-1}$	$\rho_0 [1 - (q_B - 1)\beta H_{\mathcal{O}}]^{1/(q_B-1)}$	$\rho_0 \exp(-\beta H_{\mathcal{O}})$
Entropy	$S = k \left( \frac{q_{\mathcal{O}}}{q_{\mathcal{O}}-1} \right) - k \ln[\rho_0]$		



The analysis presented here aimed at the additivity of the entropy within thermodynamic systems and it was not intended to contest the validity of Eq. (2), which is applicable for the original Tsallis entropy  $S_{\text{TS}}$  under joint observation of independent random variables. It was also seen that when weakly combining systems the parameter  $q$  varies with the dimensionality of the system's phase space following the additive rule for weak interactions (Eq. (12)).

## V. ACKNOWLEDGMENTS

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